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Manuals in Orthogonal Categories

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The notion of a manual, introduced and investigated in detail by Foulis, Randall, and their followers, has turned out to be further reaching than its originators had envisaged. Its omnipresence is comparable with the notion of a sheaf, whose significance is well recognized by every pure and applied mathematician. The principal concern of this paper is to give an appropriate vehicle, as general as possible, by which the theory of manuals can be developed. The vehicle is called an orthogonal category, which is akin to the notion of a category with coproducts. Orthogonal categories provide also a new perspective on the notion of a sheaf over a complete Boolean algebra, deepening our comprehension of Boolean mathematics and paving the way to quantum mathematics.

INTRODUCTION

The notion of a manual has been investigated by Foulis and Randall's school as a vehicle for studying the operational foundations of empirical sciences. Category theory has enabled us to liberalize the notion so as to cover a wider sphere of mathematics ranging from algebraic geometry to functional analysis, for which the reader is referred to Nishimura (1993*b*, 1994, n.d.-*a,b*). The omnipresence of manuals tempted us to build a unifying framework for the theory of manuals. The resulting structure, which we present in this paper to this end, is called an *orthogonal category*. Orthogonal categories are very akin to categories with coproducts, but neither sort of category is subsumed by the other. The motivating model of orthogonal categories has been the dual category $\mathfrak{Hloc}$ of the category $\mathfrak{Hil}$ of complex Hilbert spaces and contractive linear mappings, in which orthogonal sums are by no means coproducts.

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The principal concern of this paper is to show that the theory of manuals can and should be developed within the framework of orthogonal categories, which is the subject of Section 2. Orthogonal categories also bring forth a new insight into sheaves over complete Boolean algebras. Since Boolean mathematics is concerned with sheaves over a complete Boolean algebra, this deepens our understanding of Boolean mathematics, which is the subject of Section 3. We will see in a subsequent paper (Nishimura, n.d.-*c*) that Boolean mathematics within the framework of orthogonal categories affords a quantum generalization of sheaves. Section 1 is devoted to a review of Boolean-valued set theory.

In this paper a Hilbert space always means a complex Hilbert space. Given a complete Boolean algebra $\mathbf{B}$ and an element p of $\mathbf{B}$, the subset $\{q \in \mathbf{B} \mid q \leq p\}$ can naturally be regarded as a complete Boolean algebra, called the *relative algebra* of $\mathbf{B}$ with respect to p and denoted by $\mathbf{B} \mid p$. The set R of real numbers is usually regarded as a metric space with respect to the metric $d(x, y) = |x-y|$ $(x, y \in R)$. A *diagram* in a category $\mathfrak{K}$ is a functor from a small category $\mathfrak{I}$ (called the *indexing category*) to $\mathfrak{K}$. If the indexing category is a cone from a discrete category Λ, the suggestive notation $\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}_\lambda} \mathbf{Y}\}_{\lambda \in \Lambda}$ is used for diagrams.

1. BOOLEAN-VALUED SET THEORY

Let us quickly review the rudiments of Boolean-valued set theory. Let $\mathbf{B}$ be a complete Boolean algebra, which shall be fixed throughout this section. We define $V_\alpha^{(\mathbf{B})}$ by transfinite induction on ordinal α as follows:

$$V_0^{(\mathbf{B})} = \emptyset \tag{1.1}$$

$$V_\alpha^{(\mathbf{B})} = \{u \mid u\colon \mathfrak{D}(u) \to \mathbf{B} \text{ and } \mathfrak{D}(u) \subset \bigcup_{\beta<\alpha} V_\beta^{(\mathbf{B})} \tag{1.2}$$

Then the Boolean-valued universe $V^{(\mathbf{B})}$ of Scott and Solovay is defined as follows:

$$V^{(\mathbf{B})} = \bigcup_{\alpha \in \mathrm{On}} V_\alpha^{(\mathbf{B})} \tag{1.3}$$

where On is the class of all ordinal numbers. The class $V^{(\mathbf{B})}$ can be considered to be a Boolean-valued model of set theory by defining $[\![u \in v]\!]$ and $[\![u = v]\!]$ for $u, v \in V^{(\mathbf{B})}$ with simultaneous induction

$$[\![u \in v]\!] = \sup_{y \in \mathfrak{D}(v)} (v(y) \wedge [\![u = y]\!]) \tag{1.4}$$

$$[\![u = v]\!] = \inf_{x \in \mathfrak{D}(u)} (u(x) \to [\![x \in v]\!]) \wedge \inf_{y \in \mathfrak{D}(v)} (v(y) \to [\![y \in u]\!]) \tag{1.5}$$

and by assigning a Boolean value $[\![\Theta]\!]$ to each formula Θ without free variables inductively as follows:

$$[\![\neg\Theta]\!] = \neg[\![\Theta]\!] \tag{1.6}$$

$$[\![\Theta_1 \vee \Theta_2]\!] = [\![\Theta_1]\!] \vee [\![\Theta_2]\!] \tag{1.7}$$

$$[\![\Theta_1 \wedge \Theta_2]\!] = [\![\Theta_1]\!] \wedge [\![\Theta_2]\!] \tag{1.8}$$

$$[\![\forall x\Theta(x)]\!] = \inf_{u\in V^{(\mathbf{B})}} [\![\Theta(u)]\!] \tag{1.9}$$

$$[\![\exists x\Theta(x)]\!] = \sup_{u\in V^{(\mathbf{B})}} [\![\Theta(u)]\!] \tag{1.10}$$

Every theorem of standard mathematics, no matter what branch it belongs to, can be regarded in principle as a theorem of the Zermelo–Fraenkel set theory with the axiom of choice, usually abbreviated to ZFC. The following theorem gives a powerful transfer principle from standard mathematics to Boolean mathematics.

Theorem 1.1. If Θ is a theorem of ZFC, then so is $[\![\Theta]\!] = 1$.

The class V of all sets can be embedded into $V^{(\mathbf{B})}$ by transfinite induction as follows:

$$\check{y} = \{(\check{x}, 1) | x \in y\} \qquad \text{for} \quad y \in V$$

Proposition 1.2. For $x, y \in V$, we have:

(1) $[\![\check{x} \in \check{y}]\!] = \begin{cases} 1 & \text{if} \quad x \in y \\ 0 & \text{otherwise} \end{cases}$

(2) $[\![\check{x} = \check{y}]\!] = \begin{cases} 1 & \text{if} \quad x = y \\ 0 & \text{otherwise} \end{cases}$

A (possibly empty) family $\{p_\lambda\}_{\lambda\in\Lambda}$ of nonzero elements of $\mathbf{B}$ is called a *partial partition of unity of* $\mathbf{B}$ if $p_\lambda \wedge p_{\lambda'} = 0$ for any $\lambda \neq \lambda'$. A partial partition of unity $\{p_\lambda\}_{\lambda\in\Lambda}$ of $\mathbf{B}$ is called a *partition of unity* of $\mathbf{B}$ if $\vee_{\lambda\in\Lambda} p_\lambda = 1$.

As remarked in Nishimura (1993*b*, p. 1297), every poset and so the complete Boolean algebra $\mathbf{B}$ in particular can be regarded as a category. The objects of the category $\mathbf{B}$ are the elements of $\mathbf{B}$. Given a pair (p, q) of objects of the category $\mathbf{B}$, it is always the case that there exists at most one arrow from p to q, and there is one iff $p \leq q$. We denote by $\mathfrak{Ens}$ the category of sets and functions. A *presheaf* on $\mathbf{B}$ is a contravariant functor $\mathcal{F}$ from the category $\mathbf{B}$ to the category $\mathfrak{Ens}$, in which, given $p, q \in \mathbf{B}$ and $x \in \mathcal{F}(\mathbf{q})$ with $p \leq q$ so that there exists a unique arrow $f_{q,p}$ from p to q in the category $\mathbf{B}$, we often write $\mathcal{F}_{p,q}(x)$ for $\mathcal{F}(f_{q,p})(x)$. A presheaf $\mathcal{F}$ on $\mathbf{B}$ is called a *sheaf* on **B** if for any partial partition $\{p_\lambda\}_{\lambda\in\Lambda}$ of unity of **B** and any family $\{x_\lambda\}_{\lambda\in\Lambda}$ with $x_\lambda \in \mathcal{F}(p_\lambda)$ for each $\lambda \in \Lambda$, there exists a unique $x \in \mathcal{F}(\vee_{\lambda\in\Lambda}p_\lambda)$ with $\mathcal{F}_{p_\lambda, \vee_{\lambda\in\Lambda}p_\lambda}(x) = x_\lambda$ for each $\lambda \in \Lambda$. Our definition of a sheaf appears to diverge from the standard one, but the following theorem will take us back to the conventional one.

Theorem 1.3. Let $\mathcal{F}$ be a sheaf on **B**. For any family $\{p_\lambda\}_{\lambda\in\Lambda}$ of elements p_λ in **B** and any family $\{x_\lambda\}_{\lambda\in\Lambda}$ with $x_\lambda \in \mathcal{F}(p_\lambda)$ for each $\lambda \in \Lambda$, if

$$\mathcal{F}_{p_\lambda \wedge p_{\lambda'}, p_\lambda}(x_\lambda) = \mathcal{F}_{p_\lambda \wedge p_{\lambda'}, p_{\lambda'}}(x_{\lambda'}) \qquad \text{for any} \quad \lambda, \lambda' \in \Lambda$$

then there exists a unique $x \in \mathcal{F}(\vee_{\lambda\in\Lambda}p_\lambda)$ with $\mathcal{F}_{p_\lambda, \vee_{\lambda\in\Lambda}p_\lambda}(x) = x_\lambda$ for each $\lambda \in \Lambda$.

Proof. By Zermelo's well-ordering theorem, we can assume that Λ is the set of all the ordinal numbers α less than some ordinal number α_0. Let $q_\alpha = p_\alpha \wedge \rceil \vee_{\beta<\alpha} p_\beta$ for each ordinal number $\alpha < \alpha_0$. It is easy to see that the family $\{q_\alpha\}_{\alpha<\alpha_0}$ is a partial partition of unity of **B** and $\vee_{\beta<\alpha} q_\beta = \vee_{\beta<\alpha} p_\beta$ for each $\alpha < \alpha_0$. Let $y_\alpha = \mathcal{F}_{q_\alpha, p_\alpha}(x_\alpha)$ for any $\alpha < \alpha_0$. Then there exists a unique $y \in \mathcal{F}(\vee_{\alpha<\alpha_0} q_\alpha)$ with

$$\mathcal{F}_{q_\alpha, \vee_{\alpha<\alpha_0} q_\alpha}(y) = y_\alpha \qquad \text{for any} \quad \alpha < \alpha_0$$

If $x \in \mathcal{F}(\vee_{\alpha<\alpha_0} p_a)$ is such that

$$\mathcal{F}_{p_\alpha, \vee_{\alpha<\alpha_0} p_\alpha}(x) = x_\alpha \qquad \text{for any} \quad \alpha < \alpha_0$$

then

$$\mathcal{F}_{q_\alpha, \vee_{\alpha<\alpha_0} q_\alpha}(x) = y_\alpha \qquad \text{for any} \quad \alpha < \alpha_0$$

so that $x = y$, which has just established the uniqueness part of the theorem. To show the existence part of the theorem, it suffices to see that

$$\mathcal{F}_{p_a, \vee_{\alpha<\alpha_0} p_\alpha}(y) = x_\alpha \qquad \text{for each} \quad \alpha < \alpha_0$$

which may be established as follows. Let α, β be ordinal numbers with $\beta < \alpha < \alpha_0$. Then we have

$$\begin{aligned}
\mathcal{F}_{p_\alpha \wedge q_\beta, \vee_{\alpha<\alpha_0} p_\alpha}(y) &= \mathcal{F}_{p_\alpha \wedge q_\beta, q_b}(\mathcal{F}_{q_\beta, \vee_{\alpha<\alpha_0} p_\alpha}(y)) \\
&= \mathcal{F}_{p_\alpha \wedge q_\beta, q_\beta}(y_\beta) \\
&= \mathcal{F}_{p_\alpha \wedge q_\beta, q_\beta}(\mathcal{F}_{q_\beta, p_\beta}(x_\beta)) \\
&= \mathcal{F}_{p_\alpha \wedge q_\beta, p_\beta}(x_\beta) \\
&= \mathcal{F}_{p_\alpha \wedge q_\beta, p_\alpha \wedge p_\beta}(\mathcal{F}_{p_\alpha \wedge p_\beta, p_\beta}(x_\beta)) \\
&= \mathcal{F}_{p_\alpha \wedge q_\beta, p_\alpha \wedge p_\beta}(\mathcal{F}_{p_\alpha \wedge p_\beta, p_\alpha}(x_\alpha)) \\
&= \mathcal{F}_{p_\alpha \wedge q_\beta, p_\alpha}(x_\alpha)
\end{aligned}$$

Since the family $\{p_\alpha \wedge q_\beta\}_{\beta<\alpha} \cup \{q_\alpha\}$ is a partial partition of unity of **B** whose supremum is p_α, we have

$$\mathcal{F}_{p_\alpha, \vee_{\alpha<\alpha_0} q_\alpha}(y) = x_\alpha$$

As α was an arbitrary ordinal number with $\alpha < \alpha_0$, the proof is complete. ■

Given presheaves $\mathcal{F}$, $\mathcal{G}$ on **B**, a *morphism of presheaves* from $\mathcal{F}$ to $\mathcal{G}$ is a natural transformation τ from the functor $\mathcal{F}$ to the functor $\mathcal{G}$. In particular, if $\mathcal{F}$ and $\mathcal{G}$ happen to be sheaves on **B**, a morphism of presheaves from $\mathcal{F}$ to $\mathcal{G}$ is also called a *morphism of sheaves* from $\mathcal{F}$ to $\mathcal{G}$.

Given presheaves $\mathcal{F}$, $\mathcal{G}$ on **B**, if $\mathcal{F}(p)$ is a subset of $\mathcal{G}(p)$ for each $p \in$ **B** and $\mathcal{F}_{p,q}$ is the restriction of $\mathcal{G}_{p,q}$ for any p, $q \in$ **B** with $p \leq q$, then $\mathcal{F}$ is called a *subpresheaf* of $\mathcal{G}$. In particular, if $\mathcal{F}$ and $\mathcal{G}$ happen to be sheaves on **B**, then $\mathcal{F}$ is called a *subsheaf* of $\mathcal{G}$.

Given presheaves $\mathcal{F}$, $\mathcal{G}$ on **B**, the presheaf which assigns to each $p \in$ **B** $\mathcal{F}(p) \times \mathcal{G}(p)$ and which assigns to each arrow $p \to q$ the function

$$(x, y) \in \mathcal{F}(q) \times \mathcal{G}(q) \mapsto (\mathcal{F}_{p,q}(x), \mathcal{G}_{p,q}(y)) \in \mathcal{F}(p) \times \mathcal{G}(p)$$

is denoted by $\mathcal{F} \times_{\mathbf{B}} \mathcal{G}$. A morphism of presheaves of $\mathcal{F} \times_{\mathbf{B}} \mathcal{G}$ can be regarded as a morphism of two arguments. The discussion can be generalized easily to several arguments.

Given $u \in V^{(\mathbf{B})}$, we are going to build its associated shear $\hat{u}$ on **B**. Each $p \in$ **B** determines an equivalence relation $\equiv_p$ on $V^{(\mathbf{B})}$ as follows:

$$v \equiv_p w \qquad \text{iff} \quad [\![v = w]\!] \geq p \tag{1.11}$$

For each $v \in V^{(\mathbf{B})}$ we write $[v]_p$ for the equivalence class of v with respect to the equivalence relation $\equiv_p$. For each $p \in$ **B** we write $V^{(\mathbf{B})}(p, u)$ for the class $\{v \in V^{(\mathbf{B})} | [\![v \in u]\!] \geq p\}$. We write $V^{(\mathbf{B})}[p, u]$ for the set $\{[v]_p | v \in V^{(\mathbf{B})}(p, u)\}$. We define a presheaf $\hat{u}$ on **B** as follows:

$$\hat{u}(p) = V^{(\mathbf{B})}[p, u] \qquad \text{for each} \quad p \in \mathbf{B} \tag{1.12}$$

$$\hat{u}_{p,q}([v]_q) = [v]_p \qquad \text{for any} \quad p, q \in \mathbf{B} \quad \text{with} \quad p \leq q \tag{1.13}$$

It is not difficult to see that this is indeed well defined and besides that $\hat{u}$ is a sheaf on **B**.

Let φ: $u \to v$ be a function in $V^{(\mathbf{B})}$. We are going to construct its associated morphism $\hat{\varphi}$ of sheaves from $\hat{u}$ to $\hat{v}$. For each $p \in$ **B** we define a function $\hat{\varphi}_p$ from $\hat{u}(p)$ to $\hat{v}(p)$ as follows:

$$\hat{\varphi}_p([w]_p) = [\varphi(w)]_p \qquad \text{for each} \quad w \in V^{(\mathbf{B})}(p, u) \tag{1.14}$$

It is not difficult to see that this is indeed well defined and besides that the assignment $p \in \mathbf{B} \mapsto \hat{\varphi}_p$, denoted by $\hat{\varphi}$, is a morphism of sheaves from $\hat{u}$ to $\hat{v}$.

Conversely, given a presheaf $\mathscr{F}$ on $\mathbf{B}$, we are going to build its associated element $\tilde{\mathscr{F}}$ of $V^{(\mathbf{B})}$. First we define an element $\check{\mathscr{F}}$ of $V^{(\mathbf{B})}$ as follows:

$$\check{\mathscr{F}} = \{(\check{x}, p) | p \in \mathbf{B} \text{ and } x \in \mathscr{F}(p)\} \tag{1.15}$$

It is not difficult to see that $\sim_{\mathscr{F}} = \{((x, y)^{\vee}, r) | x \in \mathscr{F}(p)$ for some $p \in \mathbf{B}$, $y \in \mathscr{F}(q)$ for some $q \in \mathbf{B}$, and $r = \sup\{r' \in \mathbf{B} | r' \leq p \wedge q$ and $\mathscr{F}_{r',p}(x) = \mathscr{F}_{r',q}(y)\}\}$ is an equivalence relation on $\check{\mathscr{F}}$ in $V^{(\mathbf{B})}$. The quotient set of $\check{\mathscr{F}}$ with respect to this equivalence relation in $V^{(\mathbf{B})}$ is denoted by $\tilde{\mathscr{F}}$.

Given a morphism τ: $\mathscr{F} \to \mathscr{G}$ of presheaves on $\mathbf{B}$ from $\mathscr{F}$ to $\mathscr{G}$, we are going to construct its associated function $\tilde{\tau}$ from $\tilde{\mathscr{F}}$ to $\tilde{\mathscr{G}}$ in $V^{(\mathbf{B})}$. First we define a function $\check{\tau}$: $\check{\mathscr{F}} \to \check{\mathscr{G}}$ in $V^{(\mathbf{B})}$ as follows:

$$\check{\tau} = \{((x, \tau_p(x))^{\vee}, p) | p \in \mathbf{B} \text{ and } x \in \mathscr{F}(p)\} \tag{1.16}$$

Since the function $\check{\tau}$ respects the equivalence relations $\sim_{\mathscr{F}}$ and $\sim_{\mathscr{G}}$ in $V^{(\mathbf{B})}$, it naturally brings forth a function $\tilde{\tau}$: $\tilde{\mathscr{F}} \to \tilde{\mathscr{G}}$ in $V^{(\mathbf{B})}$. The construction of $\tilde{\tau}$ from τ can be generalized easily to functions of several arguments.

We denote by $\mathfrak{Ens}^{(\mathbf{B})}$ the category whose objects are all elements of $V^{(\mathbf{B})}$ and whose morphisms are all functions in $V^{(\mathbf{B})}$. We denote by $\mathbb{Sh}_{\mathbf{B}}$ the category whose objects are all sheaves on $\mathbf{B}$ and whose morphisms are all morphisms of sheaves on $\mathbf{B}$. By tidying up the preceding discussions, it is not difficult to see the following result.

Theorem 1.4. The categories $\mathfrak{Ens}^{(\mathbf{B})}$ and $\mathbb{Sh}^{\mathbf{B}}$ are equivalent.

Since the Scott–Solovay universe $V^{(\mathbf{B})}$ enjoys ZFC, we can construct the set $R_{\mathbf{B}}$ of real numbers in $V^{(\mathbf{B})}$ by any one of the well-known methods, for which the reader is referred to Takeuti (1978). We denote by Ω the Stonean space of $\mathbf{B}$ and by Ω_p the Stonean space of $\mathbf{B} | p$ for each $p \in \mathbf{B}$. Note that if $p \leq q$ in $\mathbf{B}$, then Ω_p can naturally be regarded as a topological subspace of Ω_p. The corresponding sheaf of $R_{\mathbf{B}}$ can be represented by the sheaf $\mathscr{R}_{\mathbf{B}}$ on $\mathbf{B}$, which assigns to each $p \in \mathbf{B}$ the set of real-valued Borel functions on Ω_p, where two real-valued Borel functions on Ω_p are identified if they differ only within a meager Borel subset of Ω_p, and which assigns to each pair (p, q) of elements of $\mathbf{B}$ with $p \leq q$ the mapping $f \in \mathscr{R}_{\mathbf{B}}(q) \mapsto f | \Omega_p$ (the restriction of f to Ω_p). We denote by $\mathscr{R}_{\mathbf{B}}^{\infty}$ the subpresheaf of $\mathscr{R}_{\mathbf{B}}$ such that $\mathscr{R}_{\mathbf{B}}^{\infty}(p)$ consists of all essentially bounded, real-valued Borel functions on Ω_p for each $p \in \mathbf{B}$, where a function on Ω_p is called essentially bounded if there exists a positive number r such that $|f(x)| \leq r$ for any $x \in \Omega_p$ except some meager Borel subset of Ω_p. We denote by $\mathscr{R}_{\mathbf{B}}^{0}$ the subpresheaf of $\mathscr{R}_{\mathbf{B}}$ such that $\mathscr{R}_{\mathbf{B}}^{0}(p)$ consists of all constant functions on Ω_p for each $p \in \mathbf{B}$.

2. ORTHOGONAL CATEGORIES

Let us begin this section with a brief review on a version of manuals of Boolean locales. The category of complete Boolean algebras and complete Boolean homomorphisms is denoted by $\mathfrak{Bool}$. The dual category of $\mathfrak{Bool}$ is denoted by $\mathfrak{BLoc}$. Its objects are called *Boolean locales*. If we regard a Boolean locale $\mathbf{X}$ as an object in $\mathfrak{Bool}$, it is often denoted by $\mathcal{P}(\mathbf{X})$ for emphasis, though $\mathbf{X}$ and $\mathcal{P}(\mathbf{X})$ denote the same entity. The opposite $\mathbf{f}^{\mathrm{op}}$ of a morphism $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ in $\mathfrak{BLoc}$, which is a complete Boolean homomorphism from $\mathcal{P}(\mathbf{Y})$ to $\mathcal{P}(\mathbf{X})$, is usually denoted by $\mathcal{P}(\mathbf{f})$. A morphism $\mathbf{f}$ of $\mathfrak{BLoc}$ is called an *embedding* if $\mathcal{P}(\mathbf{f})$ is surjective. Two embeddings $\mathbf{f}: \mathbf{Y} \to \mathbf{X}$ and $\mathbf{g}: \mathbf{Z} \to \mathbf{X}$ with the same codomain are said to be *equivalent* if there exists an isomorphism $\mathbf{h}: \mathbf{Y} \to \mathbf{Z}$ in $\mathfrak{BLoc}$ such that $\mathbf{f} = \mathbf{g} \circ \mathbf{h}$. Given a Boolean locale $\mathbf{X}$ and $x \in \mathcal{P}(\mathbf{X})$, the morphism $\mathbf{i}_x: \mathbf{X}|x \to \mathbf{X}$ is an embedding, where $\mathcal{P}(\mathbf{X}|x) = \mathcal{P}(\mathbf{X})|x$ and $\mathcal{P}(\mathbf{i}_x)(y) = x \wedge y$ for each $y \in \mathcal{P}(\mathbf{X})$. Any embedding into $\mathbf{X}$ is equivalent to $\mathbf{i}_x$ for a unique $x \in \mathcal{P}(\mathbf{X})$. A Boolean locale $\mathbf{X}$ is called *trivial* if $\mathcal{P}(\mathbf{X})$ is a trivial Boolean algebra, i.e., if $\mathcal{P}(\mathbf{X})$ consists of a single element. Since the category $\mathfrak{Bool}$ is complete, the category $\mathfrak{BLoc}$ is cocomplete.

Let $\mathfrak{M}$ be a small subcategory of the category $\mathfrak{BLoc}$. A diagram of $\mathfrak{BLoc}$ is said to be *in* $\mathfrak{M}$ if all the objects and morphisms occurring in the diagram lie in $\mathfrak{M}$. Boolean locales $\mathbf{X}$ and $\mathbf{Y}$ in $\mathfrak{M}$ are said to be *$\mathfrak{M}$-orthogonal*, in notation $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$, if there exists a coproduct diagram $\mathbf{X} \xrightarrow{\mathbf{f}} \mathbf{Z} \xleftarrow{\mathbf{g}} \mathbf{Y}$ of $\mathfrak{BLoc}$ lying in $\mathfrak{M}$. A Boolean locale $\mathbf{X}$ in $\mathfrak{M}$ is said to be *$\mathfrak{M}$-maximal* if for any Boolean locale $\mathbf{Y}$ in $\mathfrak{M}$, $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$ implies that $\mathbf{Y}$ is trivial. Boolean locales $\mathbf{X}$ and $\mathbf{Y}$ in $\mathfrak{M}$ are said to be *$\mathfrak{M}$-equivalent*, in notation $\mathbf{X} \approx_{\mathfrak{M}} \mathbf{Y}$, provided that for any Boolean locale $\mathbf{Z}$ in $\mathfrak{M}$, $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Z}$ iff $\mathbf{Y} \perp_{\mathfrak{M}} \mathbf{Z}$. Obviously $\mathfrak{M}$-equivalence is an equivalence relation among the Boolean locales in $\mathfrak{M}$. We denote by $[\mathbf{X}]_{\mathfrak{M}}$ the equivalence class of $\mathbf{X}$ with respect to $\mathfrak{M}$-equivalence. A coproduct diagram $\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}_\lambda} \mathbf{X}\}_{\lambda \in \Lambda}$ of $\mathfrak{BLoc}$ lying in $\mathfrak{M}$ is called an *$\mathfrak{M}$-coproduct diagram* if for any coproduct diagram

$$\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}'_\lambda} \mathbf{X}'\}_{\lambda \in \Lambda}$$

of $\mathfrak{BLoc}$ lying in $\mathfrak{M}$, the unique morphism $\mathbf{g}: \mathbf{X} \to \mathbf{X}'$ of $\mathfrak{BLoc}$ with $\mathbf{g} \circ \mathbf{f}_\lambda = \mathbf{f}'_\lambda$ for any $\lambda \in \Lambda$ belongs to $\mathfrak{M}$, in which $\mathbf{X}$ is called an *$\mathfrak{M}$-coproduct* of $\mathbf{X}_\lambda$'s and is denoted by $\Sigma_{\lambda \in \Lambda} \oplus \mathbf{X}_\lambda$. If Λ is a finite set, say, $\Lambda = \{1, 2\}$, then such a notation as $\mathbf{X}_1 \oplus_{\mathfrak{M}} \mathbf{X}_2$ is preferred. If Λ is empty, $\mathbf{X} = \Sigma_{\lambda \in \Lambda} \oplus_{\mathfrak{M}} \mathbf{X}_\lambda$ is no other than a trivial Boolean locale which is an initial object in $\mathfrak{M}$. In this case $\mathbf{X}$ is called an *$\mathfrak{M}$-trivial* Boolean locale. An embedding $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ in $\mathfrak{M}$ is called an *$\mathfrak{M}$-embedding* if there exists an embedding $\mathbf{g}: \mathbf{Z} \to \mathbf{Y}$ in $\mathfrak{M}$ such that the diagram $\mathbf{X} \xrightarrow{\mathbf{f}} \mathbf{Y} \xleftarrow{\mathbf{g}} \mathbf{Z}$ is an $\mathfrak{M}$-coproduct diagram. In this case $\mathbf{X}$ is called an *$\mathfrak{M}$-sublocale* of $\mathbf{Y}$. Given an $\mathfrak{M}$-sublocale $\mathbf{Y}$ of a Boolean locale $\mathbf{X}$ in $\mathfrak{M}$, the $\mathfrak{M}$-embedding of $\mathbf{Y}$ into $\mathbf{X}$ is equivalent in $\mathfrak{BLoc}$ to the canonical embedding $\mathbf{i}_x$: $\mathbf{X} | x \to \mathbf{X}$ for a unique $x \in \mathcal{P}(\mathbf{X})$, in which $\mathbf{Y}$ is denoted by $\mathbf{X}_x$.

A *manual of Boolean locales* is a small subcategory $\mathfrak{M}$ of the category $\mathfrak{BLoc}$ satisfying the following conditions:

(2.1) For any pair $(\mathbf{X}, \mathbf{Y})$ of Boolean locales in $\mathfrak{M}$, there exists at most a sole morphism from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathfrak{M}$.

(2.2) There exists at least a trivial Boolean locale in $\mathfrak{M}$.

(2.3) Every trivial Boolean locale in $\mathfrak{M}$ is $\mathfrak{M}$-trivial.

(2.4) For any Boolean locales $\mathbf{X}$, $\mathbf{Y}$ in $\mathfrak{M}$, if there exists a morphism from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathfrak{M}$, then $\mathbf{Y} \perp_{\mathfrak{M}} \mathbf{Z}$ implies $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Z}$ for any Boolean locale $\mathbf{Z}$ in $\mathfrak{M}$.

(2.5) For any Boolean locales $\mathbf{X}$, $\mathbf{Y}$ in $\mathfrak{M}$ with $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$, there exists a Boolean locale $\mathbf{Z}$ of the form $\mathbf{Z} = \mathbf{X} \oplus_{\mathfrak{M}} \mathbf{Y}$.

(2.6) For any Boolean locale $\mathbf{Z}$ with $\mathbf{Z} = \mathbf{X} \oplus_{\mathfrak{M}} \mathbf{Y}$ in $\mathfrak{M}$, $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{W}$ and $\mathbf{Y} \perp_{\mathfrak{M}} \mathbf{W}$ imply $\mathbf{Z} \perp_{\mathfrak{M}} \mathbf{W}$ for any Boolean locale $\mathbf{W}$.

(2.7) For any Boolean locales $\mathbf{X}$ and $\mathbf{Y}$ in $\mathfrak{M}$, $\mathbf{X} \approx_{\mathfrak{M}} \mathbf{Y}$ iff there exists a Boolean locale $\mathbf{Z}$ in $\mathfrak{M}$ such that $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Z}$, $\mathbf{Y} \perp_{\mathfrak{M}} \mathbf{Z}$, and both of $\mathbf{X} \oplus_{\mathfrak{M}} \mathbf{Z}$ and $\mathbf{Y} \oplus_{\mathfrak{M}} \mathbf{Z}$ are $\mathfrak{M}$-maximal.

(2.8) For any commutative diagram

$$\begin{tikzcd} \mathbf{X} \arrow[rr, "\mathbf{f}"] \arrow[dr, "\mathbf{g}"'] & & \mathbf{Y} \\ & \mathbf{Z} \arrow[ur, "\mathbf{h}"'] & \end{tikzcd}$$

of $\mathfrak{BLoc}$, if $\mathbf{f}$ is in $\mathfrak{M}$ and $\mathbf{h}$ is an $\mathfrak{M}$-embedding, then $\mathbf{g}$ is in $\mathfrak{M}$.

Now we reproduce a pristine example of a manual of Boolean locales from Nishimura (1993*b*, Example 3.3), which will play a pivotal role in our future transition from Boolean mathematics to quantum mathematics.

Example 2.1. Let $\mathbf{B}$ be a complete Boolean algebra. For each $p \in \mathbf{B}$ we denote by $\mathbf{X}_p$ the Boolean locale with $\mathcal{P}(\mathbf{X}_p) = \mathbf{B} | p$. The first-class Boolean manual $\mathfrak{M}_{\mathbf{B}}$ on $\mathbf{B}$ is a subcategory of the category $\mathfrak{BLoc}$ whose objects are all $\mathbf{X}_p$ ($p \in \mathbf{B}$). A morphism $\mathbf{f}$: $\mathbf{X}_p \to \mathbf{X}_q$ of Boolean locales with $p, q \in \mathbf{B}$ lies in $\mathfrak{M}_{\mathbf{B}}$ iff $p \leq q$ and $\mathcal{P}(f)(x) = x \wedge p$ for any $x \in \mathcal{P}(\mathbf{X}_q)$. It is easy to see that the categories $\mathbf{B}$ and $\mathfrak{M}_{\mathbf{B}}$ are naturally isomorphic by the assignment $p \in \mathbf{B} \mapsto \mathbf{X}_p$.

The careful reader of Nishimura (1993*b*) might notice that two important conditions imposed on our previous notion of a manual of Boolean locales are lacking in the above list of eight conditions and also that our present definition of an $\mathfrak{M}$-coproduct diagram is a bit weaker than our previous one. They are conditions (3.7) and (3.10) of that paper. The first missing condition will be discussed within a more general context, while the remaining one that is missing is shown to be retrievable within the same general context.

So far we have constructed the notion of a manual based on the category $\mathfrak{BLoc}$. Our principal concern of this section is to build rudiments of the theory of manuals upon a category as general as possible. Obviously it would be futile to try to build such a theory upon an arbitrary category, which would lead only to general nonsense. Thus our first task is to delineate an appropriate class of categories upon which a fertile theory of manuals can be established.

A pair $(\mathfrak{K}, \mathfrak{oS}_{\mathfrak{K}})$ of a category $\mathfrak{K}$ and a class $\mathfrak{oS}_{\mathfrak{K}}$ of diagrams in $\mathfrak{K}$ is called an *orthogonal category* if it satisfies the following conditions:

(2.9) The category $\mathfrak{K}$ has an initial object.

(2.10) Every diagram in $\mathfrak{oS}_{\mathfrak{K}}$ is of the form $\{\mathbf{X} \xrightarrow{\mathbf{f}_\lambda} \mathbf{Y}\}_{\lambda\in\Lambda}$.

(2.11) For any small family $\{\mathbf{X}\}_{\lambda\in\Lambda}$ of objects in $\mathfrak{K}$ there exist an object $\mathbf{Y}$ in $\mathfrak{K}$ and a family $\{\mathbf{f}_\lambda\}_{\lambda\in\Lambda}$ of morphisms $\mathbf{f}_\lambda$: $\mathbf{X}_\lambda \to \mathbf{Y}$ in $\mathfrak{K}$ such that the diagram $\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}_\lambda} \mathbf{Y}\}_{\lambda\in\Lambda}$ lies in $\mathfrak{oS}_{\mathfrak{K}}$.

(2.12) Given a small family $\{\mathbf{X}_\lambda\}_{\lambda\in\Lambda}$ of objects in $\mathfrak{K}$, if diagrams $\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}_\lambda} \mathbf{Y}\}_{\lambda\in\Lambda}$ and $\{\mathbf{X}_\lambda \xrightarrow{\mathbf{g}_\lambda} \mathbf{Z}\}_{\lambda\in\Lambda}$ lie in $\mathfrak{oS}_{\mathfrak{K}}$, then there exists a unique morphism $\mathbf{h}$: $\mathbf{Y} \to \mathbf{Z}$ in $\mathfrak{K}$ such that $\mathbf{g}_\lambda = \mathbf{h} \circ \mathbf{f}_\lambda$ for each $\lambda \in \Lambda$.

(2.13) Given diagrams $\{\mathbf{Y}_\lambda \xrightarrow{\mathbf{g}_\lambda} \mathbf{Z}\}_{\lambda\in\Lambda}$ and $\{\mathbf{X}_\delta \xrightarrow{\mathbf{f}_\delta} \mathbf{Y}_\lambda\}_{\delta\in\Delta_\lambda}$ $(\lambda \in \Lambda)$ in $\mathfrak{K}$, the diagram $\{\mathbf{X}_\delta \xrightarrow{\mathbf{g}_\lambda\circ\mathbf{f}_\delta} \mathbf{Z} \mid \lambda \in \Lambda \text{ and } \delta \in \Delta_\lambda\}$ lies in $\mathfrak{oS}_{\mathfrak{K}}$ iff all the diagrams $\{\mathbf{Y}_\lambda \xrightarrow{\mathbf{g}_\lambda} \mathbf{Z}\}_{\lambda\in\Lambda}$ and $\{\mathbf{X}_\delta \xrightarrow{\mathbf{f}_\delta} \mathbf{Y}_\lambda\}_{\delta\in\Delta_\lambda}$ $(\lambda \in \Lambda)$ lie in $\mathfrak{oS}_{\mathfrak{K}}$, where the sets Δ_λ are assumed to be mutually disjoint.

(2.14) If a diagram $\{\mathbf{X}_\delta \xrightarrow{\mathbf{f}_\delta} \mathbf{Y} \mid \lambda \in \Lambda \text{ and } \delta \in \Delta_\lambda\}$ lies in $\mathfrak{oS}_{\mathfrak{K}}$, then there exist diagrams $\{\mathbf{X}_\delta \xrightarrow{\mathbf{g}_\delta} \mathbf{Z}_\lambda\}_{\delta\in\Delta_\lambda}$ $(\lambda \in \Lambda)$ and $\{\mathbf{Z}_\lambda \xrightarrow{\mathbf{h}_\lambda} \mathbf{Y}\}_{\lambda\in\Lambda}$ such that $\mathbf{f}_\delta = \mathbf{h}_\lambda \circ \mathbf{g}_\delta$ for any $\lambda \in \Lambda$ and any $\delta \in \Delta_\lambda$, where the sets Δ_λ are assumed to be mutually disjoint.

(2.15) If $\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}_\lambda} \mathbf{Y}\}_{\lambda\in\Lambda}$ is a diagram in $\mathfrak{K}$ and $\{\mathbf{Z}_\delta \xrightarrow{\mathbf{g}_\delta} \mathbf{Y}\}_{\delta\in\Delta}$ is also a diagram in $\mathfrak{K}$ with $\mathbf{Z}_\delta$ being an initial object of $\mathfrak{K}$ for each $\delta \in \Delta$, then the diagram $\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}_\lambda} \mathbf{Y}\}_{\lambda\in\Lambda}$ is in $\mathfrak{oS}_{\mathfrak{K}}$ iff the diagram $\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}_\lambda} \mathbf{Y}\}_{\lambda\in\Lambda} \cup \{\mathbf{Z}_\delta \xrightarrow{\mathbf{g}_\delta} \mathbf{Y}\}_{\delta\in\Delta}$ is in $\mathfrak{oS}_{\mathfrak{K}}$.

(2.16) If $\mathbf{f}$: $\mathbf{X} \to \mathbf{Y}$ is an isomorphism in $\mathfrak{K}$, then the diagram $\{\mathbf{X} \xrightarrow{\mathbf{f}} \mathbf{Y}\}$ lies in $\mathfrak{oS}_{\mathfrak{K}}$.

(2.17) Given a diagram $\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}_\lambda} \mathbf{Y}\}_{\lambda \in \Lambda}$ in $\mathfrak{os}_{\mathfrak{K}}$, if $\mathbf{f}_{\lambda_1}$ and $\mathbf{f}_{\lambda_2}$ happen to be the same morphism for some distinct $\lambda_1, \lambda_2 \in \Lambda$ (so that $\mathbf{X}_{\lambda_1} = \mathbf{X}_{\lambda_2}$), then $\mathbf{X}_{\lambda_1} = \mathbf{X}_{\lambda_2}$ is an initial object of $\mathfrak{K}$.

(2.18) If a diagram $\{\mathbf{X} \xrightarrow{\mathbf{f}} \mathbf{Y}\}$ lies in $\mathfrak{os}_{\mathfrak{K}}$, then $\mathbf{f}$ is an isomorphism.

(2.19) Given diagrams $\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}_\lambda} \mathbf{Y}\}_{\lambda \in \Lambda}$ and $\{\mathbf{X}_\delta \xrightarrow{\mathbf{g}_\delta} \mathbf{Y}\}_{\delta \in \Delta}$ in $\mathfrak{K}$, if both the diagram $\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}_\lambda} \mathbf{Y}\}_{\lambda \in \Lambda}$ and the diagram $\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}_\lambda} \mathbf{Y}\}_{\lambda \in \Lambda} \cup \{\mathbf{X}_\delta \xrightarrow{\mathbf{g}_\delta} \mathbf{Y}\}_{\delta \in \Delta}$ are in $\mathfrak{os}_{\mathfrak{K}}$, then $\mathbf{X}_\delta$ is an initial object for each $\delta \in \Delta$.

Unless confusion may arise, the category $\mathfrak{K}$ itself is called an *orthogonal category* by abuse of language. A diagram $\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}_\lambda} \mathbf{Y}\}_{\lambda \in \Lambda}$ in $\mathfrak{os}_{\mathfrak{K}}$ is called an *orthogonal sum diagram*, in which $\mathbf{Y}$ is called an *orthogonal* sum of $\mathbf{X}_\lambda$'s. Thus the class $\mathfrak{os}_{\mathfrak{K}}$ is the class of orthogonal sum diagrams in $\mathfrak{K}$. A morphism $\mathbf{f}$: $\mathbf{X} \to \mathbf{Y}$ is called an *embedding* if there exists a morphism $\mathbf{g}$: $\mathbf{Z} \to \mathbf{Y}$ in $\mathfrak{K}$ such that the diagram $\mathbf{X} \xrightarrow{\mathbf{f}} \mathbf{Y} \xleftarrow{\mathbf{g}} \mathbf{Z}$ lies in $\mathfrak{os}_{\mathfrak{K}}$. Two embeddings $\mathbf{f}$: $\mathbf{Y} \to \mathbf{X}$ and $\mathbf{g}$: $\mathbf{Z} \to \mathbf{X}$ with the same codomain are said to be *equivalent* if there exists an isomorphism $\mathbf{h}$: $\mathbf{Y} \to \mathbf{Z}$ in $\mathfrak{K}$ such that $\mathbf{f} = \mathbf{g} \circ \mathbf{h}$. An object in $\mathfrak{K}$ is called *trivial* if it is an initial object of $\mathfrak{K}$. A trivial object of $\mathfrak{K}$ can be regarded as the orthogonal sum of the empty family of objects in $\mathfrak{K}$.

With such an abstract concept as that of an orthogonal category $\mathfrak{K}$ just introduced, we feel obliged to present some examples.

Example 2.2. Let $\mathfrak{K}$ be a category with (possibly infinite) coproducts. Let $\mathfrak{cp}_{\mathfrak{K}}$ be the class of all coproduct diagrams in $\mathfrak{K}$. It often happens that the pair $(\mathfrak{K}, \mathfrak{os}_{\mathfrak{K}})$ is an orthogonal category. By way of example, the category $\mathfrak{BLoc}$ is cocomplete, and the pair $(\mathfrak{BLoc}, \mathfrak{cp}_{\mathfrak{BLoc}})$ is an orthogonal category. The category $\mathfrak{Ens}$ is complete and cocomplete, and the pairs $(\mathfrak{Ens}, \mathfrak{cp}_{\mathfrak{Ens}})$ and $(\mathfrak{SLoc}, \mathfrak{cp}_{\mathfrak{SLoc}})$ are orthogonal categories, where $\mathfrak{SLoc}$ denotes the opposite category of $\mathfrak{Ens}$, whose objects are called *set locales*. The category $\mathfrak{AbGp}$ of Abelian groups and group homomorphisms is complete and cocomplete, and the pairs $(\mathfrak{AbGp}, \mathfrak{cp}_{\mathfrak{AbGp}})$ and $(\mathfrak{AbLoc}, \mathfrak{cp}_{\mathfrak{AbLoc}})$ are orthogonal categories, where $\mathfrak{AbLoc}$ denotes the opposite category of $\mathfrak{AbGp}$, whose objects are called *Abelian locales*.

We give a special case of Example 2.2, which will be of interest later.

Example 2.3. We denote by $\mathfrak{vNA}$ the category of von Neumann algebras and normal homomorphisms, where a homomorphism of von Neumann algebras means a homomorphism of rings which is $*$-preserving (i.e., a $*$-homomorphism) and unitary (i.e., mapping the identity operator of the first von Neumann algebra to that of the second von Neumann algebra). The category $\mathfrak{vNA}$ is complete. In particular, if $\{\mathcal{M}_\lambda\}_{\lambda \in \Lambda}$ is a family of von Neumann algebras $\mathcal{M}_\lambda$ acting on Hilbert spaces $\mathcal{H}_\lambda$, then its products are isomorphic to the direct sum $\Sigma_{\lambda\in\Lambda} \oplus \mathcal{M}_\lambda$ acting on the orthogonal sum $\Sigma_{\lambda\in\Lambda} \oplus \mathcal{H}_\lambda$ [cf. p. 336 of Kadison and Ringrose (1983/1986) or §I.2.2 of Dixmier (1981)]. The dual category $\mathfrak{vNLoc}$ of $\mathfrak{vNAl}$ is cocomplete, and the pair ($\mathfrak{vNLoc}$, $\mathfrak{cp}_{\mathfrak{vNLoc}}$) is an orthogonal category, where the objects of $\mathfrak{vNLoc}$ are called von Neumann locales.

The following example shows that even if a category $\mathfrak{K}$ is cocomplete, the coproduct diagrams are not necessarily adequate to be the orthogonal diagrams.

Example 2.4. A complete Boolean algebra $\mathbf{B}$ is a poset, and can naturally be regarded as a category. It is cocomplete, but the pair ($\mathbf{B}$, $\mathfrak{cp}_{\mathbf{B}}$) is by no means an orthogonal category. The class $\mathfrak{od}_{\mathbf{B}}$ of orthogonal diagrams in $\mathbf{B}$ should be taken to be the class of diagrams $\{p_\lambda \to \vee_{\lambda\in\Lambda} p_\lambda\}_{\lambda\in\Lambda}$ in which the p_λ's are mutually disjoint. Throughout the remaining part of the paper the category $\mathbf{B}$ will be regarded as an orthogonal category in this sense.

Now we give an important example, which is not a special case of Example 2.2.

Example 2.5. We denote by $\mathfrak{Hil}$ the category of Hilbert spaces and contractive linear transformations. That is to say, a linear transformation T: $\mathcal{H} \to \mathcal{K}$ of Hilbert spaces is a morphism in $\mathfrak{Hil}$ iff $\|T(x)\| \leq \|x\|$ for any $x \in \mathcal{H}$. We take as $\mathfrak{od}_{\mathfrak{Hil}}$ the class of all diagrams $\{\mathcal{H}_\lambda \xrightarrow{U_\lambda} \mathcal{K}\}_{\lambda\in\Lambda}$ in $\mathfrak{Hil}$ such that U_λ is an isometry of $\mathcal{H}_\lambda$ into $\mathcal{K}$ for each $\lambda \in \Lambda$, the images $U_\lambda(\mathcal{H}_\lambda)$ of U_λ are mutually orthogonal in $\mathcal{K}$, and $\mathcal{K} = \Sigma_{\lambda\in\Lambda} \oplus U_\lambda(\mathcal{H}_\lambda)$. It is easy to see that ($\mathfrak{Hil}$, $\mathfrak{od}_{\mathfrak{Hil}}$) is an orthogonal category.

Example 2.5 has its dual counterpart, which has been the motivating example of our new notion of an orthogonal category.

Example 2.6. We denote by $\mathfrak{HLoc}$ the dual category of $\mathfrak{Hil}$. Its objects are called *Hilbert locales*. If a Hilbert locale $\mathbf{X}$ is regarded as an object of $\mathfrak{Hil}$, it is denoted by $\mathcal{H}(\mathbf{X})$ for emphasis, though $\mathbf{X}$ and $\mathcal{H}(\mathbf{X})$ represent the same entity. If $\mathbf{f}$ is a morphism in $\mathfrak{HLoc}$, then its dual $\mathbf{f}^{\mathrm{op}}$ is denoted by $\mathcal{H}(\mathbf{f})$. We denote by $\mathfrak{od}_{\mathfrak{HLoc}}$ the class of diagrams $\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}_\lambda} \mathbf{Y}\}_{\lambda\in\Lambda}$ in $\mathfrak{HLoc}$ such that $\mathcal{H}(\mathbf{f}_\lambda)$ is a partial isometry of $\mathcal{H}(\mathbf{Y})$ onto $\mathcal{H}(\mathbf{X}_\lambda)$ for each $\lambda \in \Lambda$, the initial spaces $\mathcal{I}(\mathcal{H}(\mathbf{f}_\lambda))$ of $\mathcal{H}(\mathbf{f}_\lambda)$ are mutually orthogonal in $\mathcal{H}(\mathbf{Y})$, and $\mathcal{H}(\mathbf{Y}) = \Sigma_{\lambda\in\Lambda} \oplus \mathcal{I}(\mathcal{H}(\mathbf{f}_\lambda))$.

Now we present some elementary propositions.

Proposition 2.7. If a diagram $\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}_\lambda} \mathbf{Y}\}_{\lambda\in\Lambda}$ lies in $\mathfrak{od}_{\mathfrak{K}}$ and a morphism $\mathbf{g}$: $\mathbf{Y} \to \mathbf{Z}$ is an isomorphism in $\mathfrak{K}$, then the diagram $\{\mathbf{X}_\lambda \xrightarrow{\mathbf{g}\circ\mathbf{f}_\lambda} \mathbf{Z}\}_{\lambda\in\Lambda}$ also lies in $\mathfrak{od}_{\mathfrak{K}}$.

Proof. Since the morphism $\mathbf{g}: \mathbf{Y} \to \mathbf{Z}$ is an isomorphism, the diagram $\{\mathbf{Y} \xrightarrow{\mathbf{f}} \mathbf{Z}\}$ lies in $\mathfrak{O}\mathfrak{S}_{\mathfrak{K}}$ by (2.16). As the diagram $\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}_\lambda} \mathbf{Y}\}$ lies in $\mathfrak{O}\mathfrak{S}_{\mathfrak{K}}$, the desired result follows from (2.13). ■

Proposition 2.8. If a diagram $\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}_\lambda} \mathbf{Y}\}_{\lambda \in \Lambda}$ lies in $\mathfrak{O}\mathfrak{S}_{\mathfrak{K}}$ and $\mathbf{g}_\lambda: \mathbf{Z}_\lambda \to \mathbf{X}_\lambda$ is an isomorphism for each $\lambda \in \Lambda$, then the diagram $\{\mathbf{Z}_\lambda \xrightarrow{\mathbf{f}_\lambda \circ \mathbf{g}_\lambda} \mathbf{Y}\}_{\lambda \in \Lambda}$ also lies in $\mathfrak{O}\mathfrak{S}_{\mathfrak{K}}$.

Proof. As in our previous proposition, this follows also from (2.13) and (2.16). ■

Let $\mathfrak{M}$ be a small subcategory of an orthogonal category $\mathfrak{K}$. A diagram in $\mathfrak{K}$ is said to be *in* $\mathfrak{M}$ if all the objects and morphisms occurring in the diagram lie in $\mathfrak{M}$. Objects $\mathbf{X}$ and $\mathbf{Y}$ of $\mathfrak{M}$ are said to be $\mathfrak{M}$*-orthogonal*, in notation $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$, if there exists an orthogonal sum diagram $\mathbf{X} \xrightarrow{\mathbf{f}} \mathbf{Z} \xleftarrow{\mathbf{g}} \mathbf{Y}$ of $\mathfrak{K}$ lying in $\mathfrak{M}$. An object of $\mathfrak{M}$ is said to be $\mathfrak{M}$*-trivial* if it is a trivial object of $\mathfrak{K}$ and also an initial object of $\mathfrak{M}$. An object $\mathbf{X}$ of $\mathfrak{M}$ is said to be $\mathfrak{M}$*-maximal* if for any object $\mathbf{Y}$ of $\mathfrak{M}$, $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$ implies that $\mathbf{Y}$ is $\mathfrak{M}$-trivial. Objects $\mathbf{X}$ and $\mathbf{Y}$ of $\mathfrak{M}$ are said to be $\mathfrak{M}$*-equivalent*, in notation $\mathbf{X} \simeq_{\mathfrak{M}} \mathbf{Y}$, provided that for any objects $\mathbf{Z}$ of $\mathfrak{M}$, $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Z}$ iff $\mathbf{Y} \perp_{\mathfrak{M}} \mathbf{Z}$. Obviously, $\mathfrak{M}$-equivalence is an equivalence relation among the objects of $\mathfrak{M}$. We denote by $[\mathbf{X}]_{\mathfrak{M}}$ the equivalence class of an object $\mathbf{X}$ of $\mathfrak{M}$ with respect to $\mathfrak{M}$-equivalence. An orthogonal sum diagram $\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}_\lambda} \mathbf{X}\}_{\lambda \in \Lambda}$ of $\mathfrak{K}$ lying in $\mathfrak{M}$ is said to be an *orthogonal* $\mathfrak{M}$*-sum diagram* if for any orthogonal sum diagram $\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}'_\lambda} \mathbf{X}'\}_{\lambda \in \Lambda}$ of $\mathfrak{K}$ lying in $\mathfrak{M}$ the unique morphism $\mathbf{g}: \mathbf{X} \to \mathbf{X}'$ of $\mathfrak{K}$ with $\mathbf{g} \circ \mathbf{f}_\lambda = \mathbf{f}'_\lambda$ for any $\lambda \in \Lambda$ belongs to $\mathfrak{M}$, in which $\mathbf{X}$ is called an *orthogonal* $\mathfrak{M}$*-sum* of $\mathbf{X}_\lambda$'s and is denoted by $\Sigma_{\lambda \in \Lambda} \oplus_{\mathfrak{M}} \mathbf{X}_\lambda$. If Λ is a finite set, say $\Lambda = \{1, 2\}$, then such a notation as $\mathbf{X}_1 \oplus_{\mathfrak{M}} \mathbf{X}_2$ is preferred. Note that an $\mathfrak{M}$-trivial object of $\mathfrak{M}$, if it exists, can be regarded as an orthogonal $\mathfrak{M}$-sum of the empty family of objects of $\mathfrak{M}$. A morphism $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ is called an $\mathfrak{M}$*-embedding* if there exists a morphism $\mathbf{g}: \mathbf{Z} \to \mathbf{Y}$ such that the diagram $\mathbf{X} \xrightarrow{\mathbf{f}} \mathbf{Y} \xleftarrow{\mathbf{g}} \mathbf{Z}$ is an orthogonal $\mathfrak{M}$-sum diagram. Given objects $\mathbf{X}$ and $\mathbf{Y}$ of $\mathfrak{M}$, if there exists an $\mathfrak{M}$-embedding $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ in $\mathfrak{M}$, then we say that $\mathbf{X}$ is an $\mathfrak{M}$*-subobject* of $\mathbf{Y}$.

Given an orthogonal category $\mathfrak{K}$, a *manual in* $\mathfrak{K}$ or a $\mathfrak{K}$*-manual* for short is a small subcategory of $\mathfrak{K}$ abiding by the following conditions:

(2.20) For any pair $(\mathbf{X}, \mathbf{Y})$ of objects in $\mathfrak{M}$, there exists at most a sole morphism from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathfrak{M}$.

(2.21) There exists at least a trivial object of $\mathfrak{K}$ in $\mathfrak{M}$.

(2.22) Every trivial object of $\mathfrak{K}$ in $\mathfrak{M}$ is $\mathfrak{M}$-trivial.

(2.23) For any objects $\mathbf{X}$, $\mathbf{Y}$ in $\mathfrak{M}$, if there exists a morphism from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathfrak{M}$, then $\mathbf{Y} \perp_{\mathfrak{M}} \mathbf{Z}$ implies $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Z}$ for any object $\mathbf{Z}$ in $\mathfrak{M}$.

(2.24) For any objects $\mathbf{X}$, $\mathbf{Y}$ in $\mathfrak{M}$ with $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$, there exists an object $\mathbf{Z}$ of the form $\mathbf{Z} = \mathbf{X} \oplus_{\mathfrak{M}} \mathbf{Y}$ in $\mathfrak{K}$.

(2.25) For any object $\mathbf{Z}$ of the form $\mathbf{Z} = \mathbf{X} \oplus_{\mathfrak{M}} \mathbf{Y}$ in $\mathfrak{M}$, $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{W}$ and $\mathbf{Y} \perp_{\mathfrak{M}} \mathbf{W}$ imply $\mathbf{Z} \perp_{\mathfrak{M}} \mathbf{W}$ for any object $\mathbf{W}$ in $\mathfrak{M}$.

(2.26) For any objects $\mathbf{X}$ and $\mathbf{Y}$ in $\mathfrak{M}$, $\mathbf{X} \simeq_{\mathfrak{M}} \mathbf{Y}$ iff there exists an object $\mathbf{Z}$ in $\mathfrak{M}$ such that $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Z}$, $\mathbf{Y} \perp_{\mathfrak{M}} \mathbf{Z}$, and both of $\mathbf{X} \oplus_{\mathfrak{M}} \mathbf{Z}$ and $\mathbf{Y} \oplus_{\mathfrak{M}} \mathbf{Z}$ are $\mathfrak{M}$-maximal.

(2.27) For any commutative diagram

$$\begin{tikzcd} \mathbf{X} \arrow[rr, "\mathbf{f}"] \arrow[dr, "\mathbf{g}"'] & & \mathbf{Y} \\ & \mathbf{Z} \arrow[ur, "\mathbf{h}"'] & \end{tikzcd}$$

of $\mathfrak{K}$, if $\mathbf{f}$ is in $\mathfrak{M}$ and $\mathbf{h}$ is an $\mathfrak{M}$-embedding, then $\mathbf{g}$ is in $\mathfrak{M}$.

A $\mathfrak{K}$-manual $\mathfrak{M}$ is said to be *rich* if it satisfies the following condition:

(2.28) For any object $\mathbf{X}$ in $\mathfrak{M}$ and any embedding $\mathbf{f}$: $\mathbf{Y} \to \mathbf{X}$ in $\mathfrak{K}$, there exists an $\mathfrak{M}$-embedding $\mathbf{f}'$: $\mathbf{Y}' \to \mathbf{X}$ in $\mathfrak{M}$ such that $\mathbf{f}$ and $\mathbf{f}'$ are equivalent in $\mathfrak{K}$.

The reader should check that our previous notion of a manual of Boolean locales is no other than that of a manual in the orthogonal category ($\mathfrak{BLoc}$, $\mathfrak{cp}_{\mathfrak{BLoc}}$).

As promised, we are now ready to show that the missing condition (3.10) of Nishimura (1993*b*) is retrievable within our general context concerning a manual $\mathfrak{M}$ in an orthogonal category $\mathfrak{K}$ as follows:

Proposition 2.9. For any object $\mathbf{X}$ in $\mathfrak{M}$, if $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{X}$, then $\mathbf{X}$ is trivial.

Proof. This follows from (2.17) and (2.20). ■

Proposition 2.10. For any finite family $\{\mathbf{X}_\lambda\}_{\lambda \in \Lambda}$ of pairwise $\mathfrak{M}$-orthogonal objects in $\mathfrak{M}$, $\Sigma_{\lambda \in \Lambda} \oplus_{\mathfrak{M}} \mathbf{X}_\lambda$ exists.

Proof. If Λ is empty, then the desired result follows from (2.21) and (2.22). If Λ consists of a single element, then the proposition is trivial. If Λ consists of two elements, then the desired result follows directly from (2.24). The general statement can be proved by induction on the number of elements in Λ by using (2.14). ■

A manual $\mathfrak{M}$ in an orthogonal category $\mathfrak{K}$ is called *σ-coherent* or *completely coherent* if it satisfies the following condition $(2.29)_\sigma$ or $(2.29)_\infty$, respectively:

$(2.29)_\sigma$ For any sequence $\{\mathbf{X}_i\}_{i\in\mathbf{N}}$ of pairwise $\mathfrak{M}$-orthogonal objects in $\mathfrak{M}$, there exists an object $\mathbf{Z}$ in $\mathfrak{M}$ such that $\mathbf{Z} = \Sigma_{i\in\mathbf{N}} \oplus_{\mathfrak{M}} \mathbf{X}_i$.

$(2.29)_\infty$ For any infinite family $\{\mathbf{X}_\lambda\}_{\lambda\in\Lambda}$ of pairwise $\mathfrak{M}$-orthogonal objects in $\mathfrak{M}$, there exists an object $\mathbf{Z}$ in $\mathfrak{M}$ with $\mathbf{Z} = \Sigma_{\lambda\in\Lambda} \oplus_{\mathfrak{M}} \mathbf{X}_\lambda$.

The discussion from Proposition 3.11 through Theorem 3.17 in Nishimura (1993*b*) still hold within our more general context of a manual $\mathfrak{M}$ in an orthogonal category $\mathfrak{K}$. Thus the manual $\mathfrak{M}$ has the *associated orthomodular poset* $\mathfrak{L}(\mathfrak{M}) = (L_{\mathfrak{M}}, \leq_{\mathfrak{M}}, \urcorner_{\mathfrak{M}}, 0_{\mathfrak{M}}, 1_{\mathfrak{M}})$, where:

(2.30) $L_{\mathfrak{M}} = \{[\mathbf{X}]_{\mathfrak{M}} \mid \mathbf{X}$ is an object of $\mathfrak{M}\}$.

(2.31) $[\mathbf{X}]_{\mathfrak{M}} \leq_{\mathfrak{M}} [\mathbf{Y}]_{\mathfrak{M}}$ iff there exists an object $\mathbf{Z}$ in $\mathfrak{M}$ such that $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Z}$ and $\mathbf{X} \oplus_{\mathfrak{M}} \mathbf{Z} \simeq_{\mathfrak{M}} \mathbf{Y}$.

(2.32) $\urcorner_{\mathfrak{M}}[\mathbf{X}]_{\mathfrak{M}} = [\mathbf{Y}]_{\mathfrak{M}}$ for an object $\mathbf{Y}$ in $\mathfrak{M}$ such that $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$ and $\mathbf{X} \oplus_{\mathfrak{M}} \mathbf{Y}$ is $\mathfrak{M}$-maximal.

(2.33) $0_{\mathfrak{M}} = [\mathbf{X}]_{\mathfrak{M}}$ for a trivial object in $\mathfrak{M}$.

(2.34) $1_{\mathfrak{M}} = [\mathbf{X}]_{\mathfrak{M}}$ for an $\mathfrak{M}$-maximal object in $\mathfrak{M}$.

Proposition 3.19 of Nishimura (1993*b*) also remains sound in our present context of a manual $\mathfrak{M}$ in an orthogonal category $\mathfrak{K}$.

Proposition 2.11. For any isomorphism $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ of $\mathfrak{H}$ lying in $\mathfrak{M}$, its inverse $\mathbf{f}^{-1}$ belongs to $\mathfrak{M}$ iff $\mathbf{f}$ is an $\mathfrak{M}$-embedding.

Now we are in a position to discuss morphisms of manuals in (possibly distinct) orthogonal categories. A *morphism* from a manual $\mathfrak{M}$ in an orthogonal category $\mathfrak{K}$ to a manual $\mathfrak{N}$ in an orthogonal category $\mathfrak{L}$ is a functor $\mathfrak{F}$ from the category $\mathfrak{M}$ to the category $\mathfrak{N}$ satisfying the following conditions:

(2.35) If $\mathbf{X}$ is trivial, then $\mathfrak{F}(\mathbf{X})$ is trivial.

(2.36) If $\mathbf{X}$ is $\mathfrak{M}$-maximal, then $\mathfrak{F}(\mathbf{X})$ is $\mathfrak{N}$-maximal.

(2.37) If $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$, then $\mathfrak{F}(\mathbf{X}) \perp_{\mathfrak{N}} \mathfrak{F}(\mathbf{Y})$ and $\mathfrak{F}(\mathbf{X} \oplus_{\mathfrak{M}} \mathbf{Y}) = \mathfrak{F}(\mathbf{X}) \oplus_{\mathfrak{N}} \mathfrak{F}(\mathbf{Y})$.

The morphism $\mathfrak{F}$ is called *σ-orthocomplete* (*orthocomplete*, resp.) if it satisfies the following condition $(2.38)_\sigma$ [$(2.38)_\infty$, resp.]:

$(2.37)_\sigma$ If $\mathbf{Y} = \Sigma_{i\in\mathbf{N}} \oplus_{\mathfrak{M}} \mathbf{X}_i$ with $\{\mathbf{X}_i\}_{i\in\mathbf{N}}$ a sequence of pairwise $\mathfrak{M}$-orthogonal objects in $\mathfrak{M}$, then $\mathfrak{F}(\mathbf{Y}) = \Sigma_{i\in\mathbf{N}} \oplus_{\mathfrak{N}} \mathfrak{F}(\mathbf{X}_i)$.

$(2.37)_\infty$ If $\mathbf{Y} = \Sigma_{\lambda\in\Lambda} \oplus_{\mathfrak{M}} \mathbf{X}_\lambda$ with $\{\mathbf{X}_\lambda\}_{\lambda\in\Lambda}$ an infinite family of pairwise $\mathfrak{M}$-orthogonal objects in $\mathfrak{M}$, then $\mathfrak{F}(\mathbf{Y}) = \Sigma_{\lambda\in\Lambda} \oplus_{\mathfrak{N}} \mathfrak{F}(\mathbf{X}_\lambda)$.

A morphism $\mathfrak{F}: \mathfrak{M} \to \mathfrak{N}$ of manuals is said to be *faithful* if for any objects $\mathbf{X}, \mathbf{Y}$ in $\mathfrak{M}$, $\mathfrak{F}(\mathbf{X}) \perp_{\mathfrak{N}} \mathfrak{F}(\mathbf{Y})$ implies $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$.

Proposition 3.20 of Nishimura (1993*b*) remains valid.

Proposition 2.12. If $\mathfrak{F}: \mathfrak{M} \to \mathfrak{N}$ is a morphism of manuals, then $\mathbf{X} \simeq_{\mathfrak{M}} \mathbf{Y}$ implies $\mathfrak{F}(\mathbf{X}) \simeq_{\mathfrak{N}} \mathfrak{F}(\mathbf{Y})$ for any objects $\mathbf{X}$, $\mathbf{Y}$ in $\mathfrak{M}$.

By this proposition we can see easily that a morphism of manuals naturally induces a homomorphism of their associated orthomodular posets. In particular, if two manuals are isomorphic, their associated orthomodular posets are isomorphic.

Let $\mathbf{B}$ be a complete Boolean algebra, which shall be fixed throughout the rest of this section. Recall that a sheaf on $\mathbf{B}$ is a contravariant functor from the category $\mathbf{B}$ to the category $\mathfrak{Ens}$ satisfying a certain mild condition. Since a contravariant functor from the category $\mathbf{B}$ to the category $\mathfrak{Ens}$ is no other than a (covariant) functor from the category $\mathbf{B}$ to the category $\mathfrak{SLoc}$, it is easy to see the following:

Proposition 2.13. A functor $\mathfrak{F}$ from the orthogonal category $(\mathbf{B}, \mathfrak{os}_{\mathbf{B}})$ to the orthogonal category $(\mathfrak{SLoc}, \mathfrak{cp}_{\mathfrak{SLoc}})$ is a sheaf iff it preserves orthogonal sum diagrams.

This proposition is suggestive of a generalization of the notion of a sheaf on $\mathbf{B}$. Given an orthogonal category $\mathfrak{K}$, a *$\mathfrak{K}$-presheaf* on $\mathbf{B}$ is a functor $\mathcal{F}$ from the category $\mathbf{B}$ to the category $\mathfrak{K}$. A $\mathfrak{K}$-presheaf $\mathcal{F}$ is called a *$\mathfrak{K}$-sheaf* if it maps orthogonal sum diagrams in $\mathbf{B}$ to orthogonal sum diagrams in $\mathfrak{K}$. A *morphism* of *$\mathfrak{K}$-presheaves* from a $\mathfrak{K}$-presheaf $\mathcal{F}$ to a $\mathfrak{K}$-presheaf $\mathcal{G}$ is a natural transformation η from the functor $\mathcal{F}$ to the functor $\mathcal{G}$. If $\mathfrak{K}$-presheaves $\mathcal{F}$ and $\mathcal{G}$ happen to be $\mathfrak{K}$-sheaves, then a morphism of $\mathfrak{K}$-presheaves from $\mathcal{F}$ to $\mathcal{G}$ is also called a *morphism of $\mathfrak{K}$-sheaves* from $\mathcal{F}$ to $\mathcal{G}$. Note that although the notion of an $\mathfrak{SLoc}$-sheaf on $\mathbf{B}$ and that of a sheaf on $\mathbf{B}$ in the previous section are essentially the same, the notion of a morphism of $\mathfrak{SLoc}$-sheaves on $\mathbf{B}$ is dual to that of a morphism of sheaves on $\mathbf{B}$ defined in the previous section. We denote by $\mathfrak{Sh}_{\mathbf{B}}(\mathfrak{K})$ the category of $\mathfrak{K}$-sheaves on $\mathbf{B}$ and their morphisms.

3. REVISITING BOOLEAN MATHEMATICS

The main concern of this section is to give a view of a Boolean-valued Hilbert space in terms of $\mathfrak{HLoc}$-sheaves on complete Boolean algebras. Suppose that we are given a complete Boolean algebra $\mathbf{B}$ with a strictly positive and almost finite measure μ on it. These entities shall be fixed throughout this section. Recall that a *measure* on $\mathbf{B}$ means a countably additive nonnegative function ν on $\mathbf{B}$ with $\nu(0) = 0$ and possibly taking the positive infinite $+\infty$ as a value. It is called *strictly positive* if $\nu(x) = 0$ implies $x = 0$ for any $x \in \mathbf{B}$. It is called *almost finite* if $\sup\{x \in \mathbf{B} \mid \nu(x) \text{ is finite}\} = 1$. Note that the strict positivity and the almost-finiteness of μ imply its complete additivity, for which the reader is referred to Tomita (1952, Lemma 5.2). Therefore our assumption of the existence of a strictly positive and almost-finite measure μ on $\mathbf{B}$ is tantamount to assuming that the Stonean space Ω of $\mathbf{B}$ is hyper-Stonean in the sense of Takesaki (1979, Definition 1.14 of Chapter III). By way of example, the projection lattice of an Abelian von Neumann algebra always satisfies this condition, for which the reader is referred to Takesaki (1979, Theorem 1.18 of Chapter III).

The aim of this section is to show that the Hilbert spaces in $V^{(\mathbf{B})}$ and the $\mathfrak{HLoc}$-sheaves on $\mathbf{B}$ are substantially the same thing from different viewpoints. Let H be a Hilbert space in $V^{(\mathbf{B})}$. Now we are going to build its associated $\mathfrak{HLoc}$-sheaf $\mathcal{F}_H$ on $\mathbf{B}$. The measure μ induces a measure μ_p on the relative subalgebra $\mathbf{B}\,|\,p$ for each $p \in \mathbf{B}$. The measure μ induces a Borel measure $\overline{\mu}$ on the Stonean space Ω of $\mathbf{B}$, and the measure μ_p induces a Borel measure $\overline{\mu}_p$ on the Stonean space Ω_p of $\mathbf{B}\,|\,p$ for each $p \in \mathbf{B}$. Note that a Borel subset of Ω_p is meager iff the measure $\overline{\mu}_p$ vanishes on it. As we have shown in Section 1, the Hilbert space H in $V^{(\mathbf{B})}$ has its associated sheaf $\hat{H}$ on $\mathbf{B}$. We define

$$\mathcal{F}_H(p) = \{x \in \hat{H}(p) \,|\, \textstyle\int \langle x, x\rangle_p^\wedge \, d\overline{\mu}_p < +\infty\} \tag{3.1}$$

for each $p \in \mathbf{B}$, where the inner product $\langle\cdot, \cdot\rangle$ of H in $V^{(\mathbf{B})}$ induces a morphism of sheaves $\langle\cdot, \cdot\rangle^\wedge$: $\hat{H} \times_{\mathbf{B}} \hat{H} \to \mathcal{R}_{\mathbf{B}}$. It is not difficult to see that $\mathcal{F}_H(p)$ is a Hilbert space with respect to the following inner product:

$$\langle x, y\rangle = \textstyle\int \langle x, y\rangle_p^\wedge \, d\overline{\mu}_p \qquad \text{for} \quad x, y \in \mathcal{F}_H(p) \tag{3.2}$$

It is easy to see that for any $p, q \in \mathbf{B}$ with $p \leq q$, the assignment $x \in \mathcal{F}_H(q) \mapsto \hat{H}_{p,q}(x)$ is a contractive linear mapping from the Hilbert space $\mathcal{F}_H(q)$ to the Hilbert space $\mathcal{F}_H(p)$, whose dual is taken to be the value of the unique arrow $p \to q$ in the category $\mathbf{B}$ under $\mathcal{F}_H$. Thus we have a functor $\mathcal{F}_H$: $\mathbf{B} \to \mathfrak{HLoc}$. It is not difficult to see the following.

Proposition 3.1. $\mathcal{F}_H$ is an $\mathfrak{HLoc}$-sheaf on $\mathbf{B}$.

Given a morphism $\mathbf{f}$: $\mathbf{X} \to \mathbf{Y}$ of $\mathfrak{HLoc}$ in $V^{(\mathbf{B})}$, the dual $\mathcal{H}(\mathbf{f})$: $\mathcal{H}(\mathbf{Y}) \to \mathcal{H}(\mathbf{X})$ of $\mathbf{f}$ is a contractive linear mapping of Hilbert spaces in $V^{(\mathbf{B})}$. Let $\mathcal{F}_{\mathbf{X}} = \mathcal{F}_{\mathcal{H}(\mathbf{X})}$ and $\mathcal{F}_{\mathbf{Y}} = \mathcal{F}_{\mathcal{H}(\mathbf{Y})}$. It is not difficult to see that its associated morphism of sheaves $\mathcal{H}(\mathbf{f})^\wedge$: $\mathcal{H}(\mathbf{Y})^\wedge \to \mathcal{H}(\mathbf{X})^\wedge$ naturally induces a contractive linear mapping of Hilbert spaces $\xi_{\mathbf{f}}^p$: $\mathcal{F}_{\mathbf{Y}}(p) \to \mathcal{F}_{\mathbf{X}}(p)$ for each $p \in \mathbf{B}$. The assignment to each $p \in \mathbf{B}$ of the dual of $\xi_{\mathbf{f}}^p$, denoted by $\tau_{\mathbf{f}}$, is a morphism of $\mathfrak{HLoc}$-sheaves on $\mathbf{B}$ from $\mathcal{F}_{\mathbf{X}}$ to $\mathcal{F}_{\mathbf{Y}}$.

Conversely, given an $\mathfrak{HLoc}$-sheaf $\mathcal{F}$ on $\mathbf{B}$, we are going to construct its associated Hilbert space $H_{\mathcal{F}}$ in $V^{(\mathbf{B})}$. Since the category $\mathfrak{HLoc}$ is a subcategory of the category $\mathfrak{AbLoc}$, $\mathcal{F}$ can be regarded as an $\mathfrak{AbLoc}$-presheaf, so that the construction of Section 1 breeds an Abelian group $\tilde{\mathcal{F}}$ in $V^{(\mathbf{B})}$. The linear structures on the $\mathcal{F}(p)$ naturally yield a morphism of presheaves ξ: $\mathcal{R}^0_{\mathbf{B}} \times_{\mathbf{B}} \mathcal{F} \to \mathcal{F}$, which makes $\tilde{\mathcal{F}}$ a module over $(\mathcal{R}^0_{\mathbf{B}})^{\sim}$ through $\tilde{\xi}$ in $V^{(\mathbf{B})}$. Note that the metric completion of $(\mathcal{R}^0_{\mathbf{B}})^{\sim}$ in $V^{(\mathbf{B})}$ is the set $R_{\mathbf{B}}$ of real numbers in $V^{(\mathbf{B})}$.

Let $p \in \mathbf{B}$ and $x, y \in \mathcal{F}(p)$. Since the functor $\mathcal{F}$: $\mathbf{B} \to \mathfrak{H}\mathfrak{L}\mathfrak{o}\mathfrak{c}$ preserves orthogonal sum diagrams, the assignment $q \in \mathbf{B}|p \mapsto \langle \mathcal{F}_{q,p}(\mathbf{x}), \mathcal{F}_{q,p}(y)\rangle$ is a completely additive function, so that by a variant of the so-called Radon–Nikodym theorem (Tomita, 1952, Theorem 4) there exists an almost unique Borel function $\eta_p(x, y)$ on Ω_p such that

$$\langle \mathcal{F}_{q,p}(x), \mathcal{F}_{q,p}(y)\rangle = \int (\mathcal{R}_{\mathbf{B}})_{q,p}(\eta_p(x, y))\, d\overline{\mu}_q \tag{3.3}$$

for every $q \in \mathbf{B}|p$. It is easy to see that the assignment $p \in \mathbf{B} \mapsto \eta_p$, denoted by η, is a morphism of sheaves from $\mathcal{F} \times_{\mathbf{B}} \mathcal{F}$ to $\mathcal{R}_{\mathbf{B}}$, so that $\tilde{\eta}$ is a real-valued function on $\tilde{\mathcal{F}} \times \tilde{\mathcal{F}}$ in $V^{(\mathbf{B})}$. It is easy to see that $\tilde{\eta}$ is a separately-additive and positive-definite binary function on $\tilde{\mathcal{F}}$ in $V^{(\mathbf{B})}$, naturally bringing forth a metric on $\tilde{\mathcal{F}}$ in $V^{(\mathbf{B})}$. The metric completion of $\tilde{\mathcal{F}}$ in $V^{(\mathbf{B})}$ is denoted by $H_{\mathcal{F}}$. The module structure of $\tilde{\mathcal{F}}$ over $(\mathcal{R}^0_{\mathbf{B}})^{\sim}$ in $V^{(\mathbf{B})}$ naturally induces a module structure of $H_{\mathcal{F}}$ over $R_{\mathbf{B}}$ in $V^{(\mathbf{B})}$.

By tidying up the preceding discussions, we have the following result.

Proposition 3.2. $H_{\mathcal{F}}$ is a Hilbert space in $V^{(\mathbf{B})}$.

Let τ: $\mathcal{F} \to \mathcal{G}$ be a morphism of $\mathfrak{H}\mathfrak{L}\mathfrak{o}\mathfrak{c}$-sheaves on $\mathbf{B}$. Let $\mathbf{X}_{\mathcal{F}} = H_{\mathcal{F}}$ and $\mathbf{X}_{\mathcal{G}} = H_{\mathcal{G}}$. We denote by $\mathcal{H}(\tau)$ the assignment $p \in \mathbf{B} \mapsto \mathcal{H}(\tau_p)$. Since $\mathcal{H}(\tau)^{\sim}$: $\tilde{\mathcal{G}} \to \tilde{\mathcal{F}}$ is a contractive mapping in $V^{(\mathbf{B})}$, it has a unique continuous extension to a function T_τ from $\mathbf{X}_{\mathcal{G}}$ to $\mathbf{X}_{\mathcal{F}}$, which is a contractive linear mapping. The dual of T_τ in $V^{(\mathbf{B})}$ is denoted by $\mathbf{f}_\tau$.

We denote by $\mathfrak{H}\mathfrak{L}\mathfrak{o}\mathfrak{c}^{(\mathbf{B})}$ the category of all objects and all morphisms of $\mathfrak{H}\mathfrak{L}\mathfrak{o}\mathfrak{c}$ in $V^{(\mathbf{B})}$. We denote by Φ the functor from $\mathfrak{H}\mathfrak{L}\mathfrak{o}\mathfrak{c}^{(\mathbf{B})}$ to $\mathfrak{S}\mathfrak{h}^{\mathbf{B}}(\mathfrak{H}\mathfrak{L}\mathfrak{o}\mathfrak{c})$ consisting of the assignments $\mathbf{X} \mapsto \mathcal{F}_{\mathbf{X}}$ and $\mathbf{f} \mapsto \tau_{\mathbf{f}}$. We denote by Ψ the functor from $\mathfrak{S}\mathfrak{h}_{\mathbf{B}}(\mathfrak{H}\mathfrak{L}\mathfrak{o}\mathfrak{c})$ to $\mathfrak{H}\mathfrak{L}\mathfrak{o}\mathfrak{c}^{(\mathbf{B})}$ consisting of the assignments $\mathcal{F} \mapsto \mathbf{X}_{\mathcal{F}}$ and $\tau \mapsto \mathbf{f}_\tau$.

It is easy, though somewhat tedious, to see the following result.

Theorem 3.3. The functor $\Psi \circ \Phi$ is naturally isomorphic to the identity functor of $\mathfrak{H}\mathfrak{L}\mathfrak{o}\mathfrak{c}^{(\mathbf{B})}$.

By using the techniques of Ozawa (1983, 1984, 1985), it is also easy, though a bit more difficult than in the preceding theorem, to see the following result.

Theorem 3.4. The functor $\Phi \circ \Psi$ is naturally isomorphic to the identity functor of $\mathfrak{S}\mathfrak{h}_{\mathbf{B}}(\mathfrak{H}\mathfrak{L}\mathfrak{o}\mathfrak{c})$.

By combining Theorems 3.4 and 3.5, we have the following.

Theorem 3.5. The categories $\mathfrak{HLoc}^{(\mathbf{B})}$ and $\mathfrak{Sh}_{\mathbf{B}}(\mathfrak{HLoc})$ are equivalent.

NOTE ADDED IN PROOF

Example 2.4 turned out to be inappropriate.

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